A Simple Targeting Technique for Two-Body Spacecraft Trajectories

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The classical orbital boundary-value problem provides a means of finding feasible orbits that pass through an initial position and a fixed target position. In this paper, a method is developed that extends the boundary-value problem and allows a choice of any combination of three of the following four target parameters: altitude, downrange angle, velocity, and flight-path angle. In addition, this method is adapted for use as a guidance algorithm for spacecraft in powered flight, and is implemented in a computer simulation of spacecraft trajectories. Two cases are studied: launch from the Earth's surface into low Earth orbit, and transfer from low Earth orbit to geosynchronous orbit. Results from the computer simulations indicate that extremely good targeting accuracy can be achieved, with resulting ΔV values very close to optimal for many cases. Overall, the technique developed herein provides a fast and straightforward means of generating a feasible trajectory that satisfies a chosen combination of target conditions. The technique is easy to implement and can be applied to a wide range of two-body, space-flight guidance problems.

Introduction

MANY spacecraft guidance algorithms exist that find exact, optimal trajectories to satisfy a given set of target conditions. These elaborate algorithms—such as those based on Pontryagin's maximum principle, Bellman's dynamic programming, Rayleigh-Ritz-type procedures, or linear tangent guidance[1]—are necessary when the exact, optimal trajectory must be found, but are too cumbersome and time-consuming to implement early in design studies when a quick, easy method for finding a feasible trajectory is needed. In addition, each algorithm is developed around one set of three target parameters among position, velocity, flight-path angle, and transfer angle, and usually cannot be used for any other combination of these parameters once developed.

Therefore, no technique exists that is easy to implement, close to optimum, and works for more than one combination of target parameters, or for more than a limited range of space-flight missions. This gap in spacecraft guidance techniques can be filled by relaxing the requirement that the trajectory be exactly optimum, and by using basic methods from orbital mechanics to ensure that the target conditions are satisfied. A means already exists, known as the orbital boundary-value problem, for finding an orbit that passes through a given initial point and a selected target point. In this paper, a technique is developed that extends the orbital boundary-value problem to allow multiple choices of target parameters. The technique is also adapted for use in a wide variety of two-dimensional space-flight targeting problems, ranging from planetary launch trajectories to orbital insertion and transfers.

Development of the Orbit-Fitting Targeting Technique

The Orbital Boundary-Value Problem

The starting point for the technique developed herein is the classical two-point orbital boundary-value problem. The problem formulation is: Given an initial position and a desired target position measured from a central body, find the feasible orbits that pass through both points as well as the required velocity vectors at the initial point that correspond to flight paths along any of the feasible transfer orbits. The technique for solving this problem has been described in detail by Battin[2] and the important results will be summarized below.

If V_{req} and V_2 are velocity vectors at P_1 and P_2, respectivly, and if they correspond to an orbit that passes through P_1 and P_2, separated by a transfer angle σ (see Fig. 1), the relationships

$$V_{1c} = V_{2c} = V_c, \quad V_{1p} = -V_{2p} = V_p$$

can easily be derived, where the subscript 1 denotes current or initial position, 2 the target or final position, c the chordal components, and p the radial components in a skewed coordinate system. The chord, C, can be found from the law of cosines, and the product of the chordal velocity component V_c and radial velocity component V_p can then be expressed in the form

$$V_c V_p = \frac{\mu C}{r_1 r_2 (1+\cos\sigma)} \tag{1}$$

This relationship is the asymptotic form of an equation for a hyperbola, with one asymptote along V_c and the second along V_p, with such a hyperbola at both the initial and target points. The hyperbolas describe the locus of feasible velocities at each point, and their properties will be exploited later in the development of the orbit-fitting targeting technique. Since the required velocity vector V_{req} (which corresponds to flight along a feasible orbit from P_1 to P_2) is not yet uniquely determined, one more parameter—such as target velocity or target flight-path angle—may be specified. Selecting either one will uniquely determine a feasible transfer orbit, and hence also a unique required velocity at the initial point. The orbital boundary-value problem, however, does not permit the designation of both target velocity and target flight-path angle as target parameters. Since for many types of missions it is desirable to specify both of these variables, a means must be found to eliminate this restriction from the boundary-value problem.

If either of the target parameters upon which the boundary-value problem is based—target radius or transfer angle—is not

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specified, the restriction is removed and any two of the remaining three unspecified variables may now be selected as target parameters. Thus, of the complete set of four parameters available—velocity, altitude, flight-path angle, and transfer angle—any three may be chosen as the target parameters, and they will uniquely determine an orbital path that passes through the initial point and achieves the desired target conditions at the target point.

Solution Technique for Any Choice of Target Parameters

When any three of the target parameters are specified and the initial position is known, the fourth target parameter can be determined as a function of the three known parameters and the initial radius. To start, the geometry at the target point is expressed in terms of the angles η and ζ, as shown in Fig. 2. The perpendicular distances A and B from the radius to the tip of V_2 and from the extension of the chord to V_2, respectively, are

$$A = V_2 \sin\eta, \qquad B = -V_2 \sin(\zeta+\eta) \tag{2}$$

A and B can also be expressed in terms of the velocity components V_c and V_p as

$$A = V_c \sin\zeta, \qquad B = V_p \sin\zeta \tag{3}$$

The locus of possible velocities at P_2 is a hyperbola [as established in Eq. (1)], with the property that the product AB is equal to a constant. Multiplying together the expressions for A and B found in Eq. (2) as well as Eq. (3), setting the resulting expressions equal to each other, substituting Eq. (1) for the product $V_c V_p$ with the chord C replaced by $C = r_1 \sin\sigma/\sin\eta$, leads to the result

$$\mu \sin\eta \tan(\sigma/2) = -r_2 V_2^2 \sin\eta \sin(\zeta+\eta) \tag{4}$$

Simple geometry, and substituting for C yields a second expression involving ζ

$$\cot\zeta = \cot\sigma - [r_2/(r_1 \sin\sigma)] \tag{5}$$

The angle ζ can be eliminated between Eqs. (4) and (5), and $\pi/2 - \phi_2$ substituted for η to finally give

$$\mu r_1(1-\cos\sigma) + r_1 r_2 V_2^2 \cos\phi_2 \cos(\phi_2 - \sigma) = r_2^2 V_2^2 \cos^2\phi_2 \tag{6}$$

This is the desired expression which relates all of the possible target parameters—altitude, velocity, flight-path angle, and transfer angle—to each other. Once any three of these parameters are specified, the fourth can be found directly from this relationship.

Altitude, Velocity, and Flight-Path Angle as Target Parameters

In this particular case, the target altitude or radius r_2, velocity V_2, and flight-path angle ϕ_2 are specified and the remaining variable, σ, must be found. To solve for σ from Eq. (6), the substitution $x = \cot(\sigma/2)$ can be used. The result is a quadratic in x of the form

$$ax^2 + bx + c = 0 \tag{7}$$

where

$$a = -r_2 V_2^2 (r_2 - r_1)\cos^2\phi_2$$

$$b = 2 r_1 r_2 V_2^2 \sin\phi_2 \cos\phi_2$$

$$c = 2\mu r_1 - r_2 V_2^2 (r_1 + r_2)\cos^2\phi_2$$

The quadratic can be solved with the standard quadratic equation, and the physical nature of the problem guarantees that both roots will be real. The smaller one will give σ corresponding to flight from P_1 to P_2, and the second root will be 180 deg further around the orbit.

Equation (6) can be used for any other combination of target parameters. However, only the case completed above will be explored further, since this case is the most useful for common space missions. (For solutions to the remaining three cases, see Ref. 3.)

Since most orbital transfers and maneuvers are intended to reach a circular orbit eventually, it is useful to specify that the transfer orbit attain the altitude of the desired circular orbit with a flight-path angle equal to zero. This means that the apogee radius of the transfer orbit is the radius of the circular orbit, with the velocity vector perpendicular to the radius. The transfer angle σ can now be determined directly by setting ϕ_2 equal to zero in Eq. (6)

$$\cos\sigma = \frac{\mu r_1 - r_2^2 V_2^2}{\mu r_1 - r_1 r_2 V_2^2} \tag{8}$$

Finding the Required Velocity

If the target velocity was not specified as a target parameter, it can be found from Eq. (6). Then the semimajor axis of the transfer orbit can be found from

$$a = \left(\frac{2}{r_2} - \frac{V_2^2}{\mu}\right)^{-1} \tag{9}$$

Since the radius at the initial point is known, the vis-viva integral can be applied at the initial point to find V_{req}. Then, substituting Eq. (9) for a and simplifying gives

$$V_{req}^2 = \frac{2\mu(r_2 - r_1)}{r_1 r_2} - V_2^2 \tag{10}$$

The angular momentum of the transfer orbit, $\boldsymbol{h}$, is defined as $\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{V}$, and its magnitude is constant. Thus, setting $\boldsymbol{h}$ at the target and initial points equal to each other leads to a solution for θ

$$\theta = \arccos\left(\frac{r_2 V_2 \cos\phi_2}{r_1 V_{req}}\right) \tag{11}$$

Equations (10) and (11) now give a simple means for finding V_{req} and θ.

Guidance Algorithm

The technique developed in the preceding section for finding the required velocity vector can easily be incorporated into a guidance algorithm for a spacecraft in powered flight. At any point in the spacecraft's trajectory, the vehicle's current position can be designated the initial point P_1 in the boundary-value problem, and the problem can be solved for

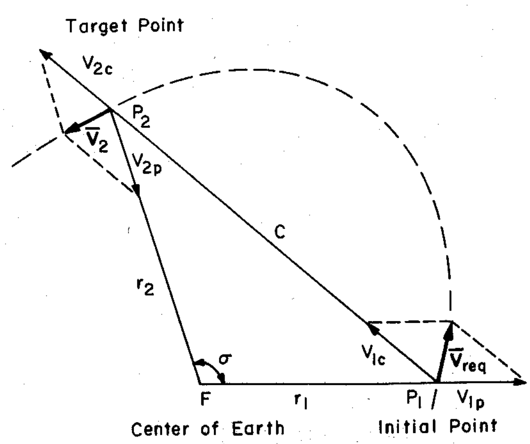

Fig. 1 Geometry of the orbital boundary-value problem.

that particular initial point and a set of target conditions at a target point. This procedure can be implemented in a computer trajectory simulation in which the vehicle's position is determined at finite time intervals by numerically integrating the equations of motion. The control variables available for use in this problem are the vehicle thrust level—most conveniently measured as a percentage of the maximum available thrust—and the thrust angle—the angle between the thrust and velocity vectors.

Determining the Velocity-to-Be-Gained

Once the required velocity for the vehicle's current position has been found, the velocity-to-be-gained V_G can be found as the vector difference between the required velocity and the vehicle's current velocity (see Fig. 3).

If ϕ is the vehicle's current flight-path angle measured from the local horizontal, and V_1 its current velocity, then V_G can be determined from the law of cosines

$$V_G^2 = V_{\text{req}}^2 + V_1^2 - 2V_{\text{req}}V_1\cos(\phi-\theta) \tag{12}$$

The angle β between V_1 and V_G can be found from the law of sines

$$\beta = \arcsin\left(\frac{V_{\text{req}}}{V_G}\sin|\phi-\theta|\right) \tag{13}$$

In many cases, β is greater than 90 deg, but the arcsin function will return values only between -90 and $+90$ deg, therefore, for the case where $V_G^2 + V_1^2 < V_{\text{req}}^2$, β must be corrected to

$$\beta = \pi - \arcsin\left(\frac{V_{\text{req}}}{V_G}\sin|\phi-\theta|\right) \tag{14}$$

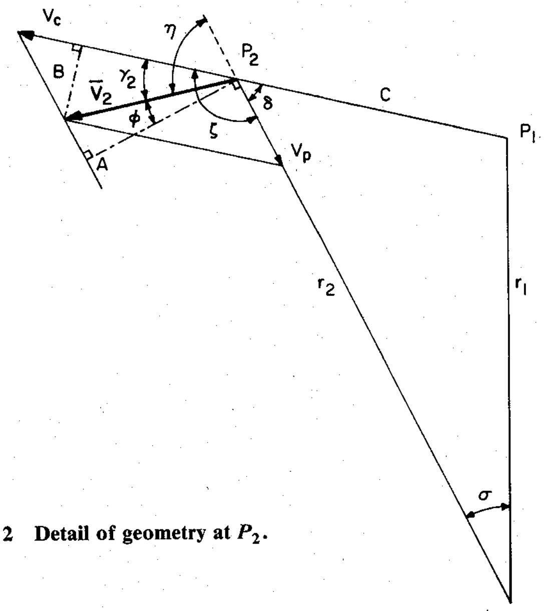

Fig. 2 Detail of geometry at P_2.

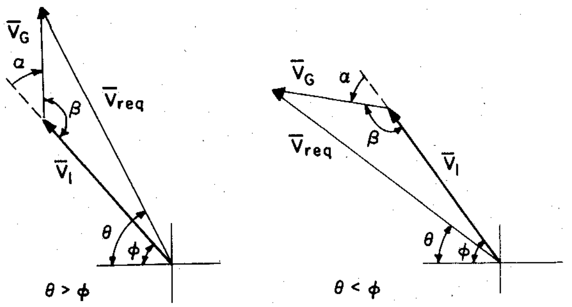

Fig. 3 Velocity-to-be-gained determination.

Determining the Thrust

For simplicity, the guidance algorithm to be used will choose the thrust vector parallel to the velocity-to-be-gained direction rather than using cross-product steering. This makes it easy to find the thrust angle α as

$$\alpha = (\pi-\beta)\operatorname{sgn}(\theta-\phi) \tag{15}$$

While the vehicle is thrusting, its velocity is approaching V_{req}. It can use maximum thrust until it reaches the point where it would overshoot the required velocity, at which point it must reduce its thrust. To determine the desired thrust level, the increment in the vehicle's velocity DV during the next time interval must be predicted, and then the thrust level that corresponds to a velocity increment less than or equal to V_G should be found. This can be done by applying the rocket equation and thrust-to-mass relationships, with the result that the desired thrust is

$$T = (mc/Dt)[1-\exp(-DV/c)] \tag{16}$$

where m is the current vehicle mass, c the exhaust velocity, and Dt the time interval.

At the point where using maximum thrust would cause DV to exceed V_G, the velocity to be gained V_G should be substituted for DV in Eq. (16) and the new thrust level used for that time interval. Once V_G becomes exactly equal to zero, the vehicle can stop thrusting entirely and coast the rest of the way to the target.

Guidance for Infeasible Target Velocities

The choice of V_2 is restricted, depending on r_1 and r_2; there being a maximum value above which no feasible orbit passes through both the initial and target points and also satisfies the target conditions. This follows since, with the selection of r_2 and V_2, the orbital semimajor axis is fixed from the vis-viva integral, and if the radius at the vehicle's current position is less than the perigee radius of the orbit determined by r_2 and V_2, then clearly no point in the orbit will pass through the vehicle's position and the problem cannot be solved. The maximum permissible value for V_2 can be found as

$$V_{2\max}^2 = \mu\left[\frac{2r_1}{r_2(r_1+r_2)}\right] \tag{17}$$

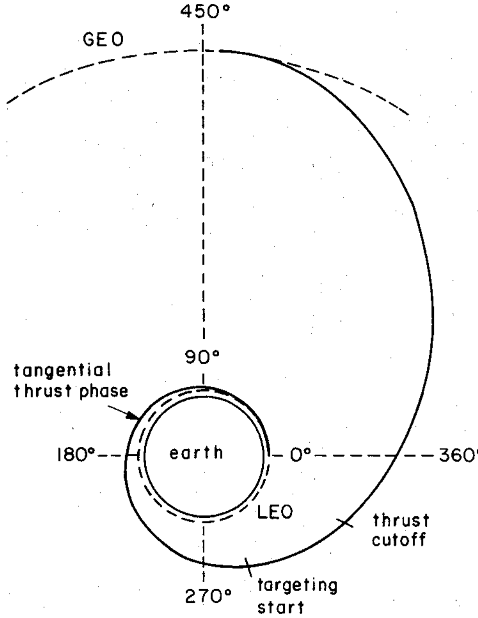

Fig. 4 Sample LEO-to-GEO transfer trajectory.

A problem arises, however, if the desired target velocity V_2 is greater than the maximum feasible target velocity V_{2max} for the vehicle at its current position. In fact, if the vehicle thrust is low and the target velocity high, it could be some time before the vehicle will attain a sufficient altitude such that V_{2max} becomes greater than V_2. During this phase of the flight a "temporary" guidance scheme must be used.

Of three possibilities tested, the best scheme would simply allow the vehicle to thrust tangentially to its flight path and follow a natural gravity-turn spiral trajectory until it has reached the altitude for which V_{2max} is greater than V_2. At that point, is has intersected the feasible transfer orbit at its perigee, and the orbit-fitting targeting procedure can be initiated.

Technique Applications and Results

Two missions were selected as test applications for the targeting technique: low Earth orbit (LEO) to geosynchronous Earth orbit (GEO) transfer, and Earth launch to LEO insertion. These cases represent two of the most frequent maneuvers in current spacecraft operations, and probably in future operations as well. In the case of LEO to GEO transfer, a near-optimum trajectory can be found fairly easily for any given thrust level, and, therefore, the performance of the technique developed in this paper can easily be compared with known near-optimum values. While it is considerably more difficult to find the optimum trajectory for the Earth launch case, a reasonable value has been well established and the results can be compared with this known value.

Simulation Technique

The trajectory simulations were performed by applying a standard fourth-order Runge-Kutta integration algorithm to the spacecraft equations of motion. The set of initial conditions—altitude, velocity, and flight-path angle—was selected according to the specific mission being analyzed. Constant thrust was used throughout most of the engine-firing phases without regard to maximum acceleration limits, although acceleration limits could be imposed if desired. Values could be chosen to limit the maximum thrust angle (hence, vehicle pitch rate) and the maximum thrust angle rate of change during the thrusting phase.

Once the vehicle has reached the target altitude, an additional burn must be performed to insert the vehicle into a circular orbit. For simplicity, and because it is the option that requires the least fuel, this burn was chosen to be impulsive in the simulations, and the required ΔV is simply the difference between the circular orbital velocity and the apogee velocity of the transfer orbit.

Orbit-to-Orbit Transfer Trajectories

The Orbital Transfer Vehicle model was selected on the basis of what eventually might be in use as part of future space activities. For reusability, liquid fuels, with an engine such as the advanced space engine, would be used, with vehicle parameters and initial and target conditions as given in Table 1. The apogee velocity of the transfer orbit was varied between 1612.6 m/s—the Hohmann transfer orbit apogee velocity—and 2110.0 m/s, depending on the thrust level. The actual target velocity was arrived at through trial and error as the one that yielded the minimum total mission ΔV for a given thrust level. For high thrust, the optimum target velocity is clearly near the Hohmann orbit apogee velocity, since the Hohmann transfer is optimal for infinite thrust capability. The optimum target velocity will increase as the thrust level decreases, since, for infinitesimal thrust, the target velocity is exactly equal to the circular velocity and the vehicle is very close to circular velocity throughout its trajectory.[4] Since the 2110 m/s target velocity led to a transfer orbit with perigee at an altitude of 6600 km, a tangential thrust trajectory was followed, as explained previously, until the vehicle reached the transfer orbit perigee altitude. After that point, the orbit-fitting targeting technique was used.

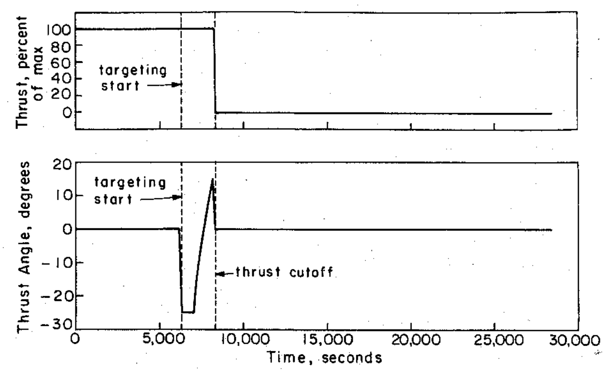

Fig. 5 Thrust, thrust angle vs time—LEO-to-GEO transfer.

Table 1 Initial and target parameters—LEO-to-GEO transfer

Specific impulse, s	450
Initial mass, kg	25,000
Initial acceleration, *g*	0.03-3.0
Initial altitude (LEO), km	372
Initial velocity, m/s	7684.5
Target altitude (GEO), km	35,863
Target velocity, m/s	1612-2110.0
Final velocity (GEO), m/s	3071.9

Table 2 ΔV comparisons for LEO-to-GEO transfer

Thrust/M_0, *g*	V_2 for minΔV, m/s	ΔV tangential thrust, m/s	ΔV targeted trajectory, m/s	Difference in ΔV, %
0.03	2110.0	4409.9	4514.6	2.49
0.10	1680.0	3998.8	4081.0	2.06
0.30	1620.0	3871.0	3897.2	0.667
1.00	1612.6	3866.8	3872.6	0.150
3.00	1612.6	3865.9	3867.1	0.031

The near-optimum trajectory to be used for performance comparisons was found by requiring the vehicle to thrust in a direction always tangential to its flight path. When the instantaneous orbit—the orbit it would be in if thrusting were terminated—had an apogee at GEO altitude, the vehicle was allowed to stop thrusting, and the required ΔV was calculated at that point.

The resulting trajectory for a sample LEO-to-GEO transfer with an initial acceleration of 0.03 *g* and a thrust angle limit of ± 25 deg is shown in polar form in Fig. 4. Figure 5 shows plots of the control variables vs time. The thrust is sustained at its maximum value until cutoff. The thrust angle moves to its lower limit as soon as the targeting is initiated, then climbs to a slightly positive value and goes to zero after thrust cutoff.

The ΔV requirement of 3554 m/s for this particular case compares favorably with the tangential-thrust near-optimum value of 3449 m/s for the thrusting phase, with a total ΔV (including the circularization burn) of 4515 m/s, compared with 4410 m/s from the tangential-thrust trajectory. These values differ by only 2.5%. The figures in Table 2 show that the technique performed well at low thrust levels, and even better at higher thrust levels, and above 1-*g* initial acceleration, the trajectory very closely resembled a Hohmann transfer ellipse—as it should, since a Hohmann transfer is optimal for high thrust levels.

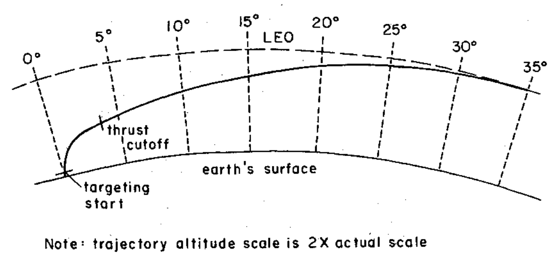

Fig. 6 Sample launch trajectory.

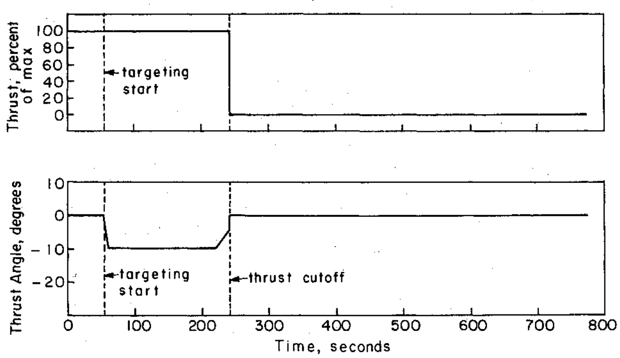

Fig. 7 Thrust, thrust angle vs time—launch to LEO.

Launch Trajectories

The launch vehicle model was chosen, for simplicity, to represent a single-stage-to-orbit vehicle with a 100,000-kg launch mass. A Space Shuttle Main Engine could be used, and the vehicle parameters, initial and target conditions, are given in Table 3.

The vehicle aerodynamic model was chosen to resemble a streamlined delta-wing Shuttle-type vehicle, with a reference area of 50 miles.[2] Since flight near the Earth's surface is dominated by atmospheric effects and it is desirable to climb out of the atmosphere as soon as possible, open-loop guidance is commonly used in the early phase of a launch and the targeting initiated only when atmospheric effects are no longer dominant. In this case, the vehicle model followed a vertical-ascent, gravity-turn trajectory until it reached an altitude of 10 km, where the atmospheric effects had diminished. The thrust angle was limited to ± 10 deg, and its maximum allowed rate of change was 2 deg/s.

The trajectory from a sample launch simulation is shown in polar form in Fig. 6. The actual target values achieved are: velocity = 7000.0 m/s, altitude = 372.002 km, and flight-path angle = -0.003 deg. This is typical of the accuracy achieved for all of the cases tested and demonstrates the basic viability of the technique. In addition, the algorithm remains relatively stable and well behaved as the vehicle approaches the target.

The behavior of the control variables is shown in Fig. 7. As soon as targeting begins, 44 s into the flight, the thrust angle decreases at the maximum allowed rate to its maximum negative value of -10 deg to pitch the vehicle down to start gaining circumferential velocity. If the thrust angle were not constrained, the vehicle would pitch over very quickly, since the transfer orbit requires a near-horizontal flight-path angle at that altitude. With the thrust angle constrained, the vehicle is permitted to pitch over only slowly, so that by the time it approaches a horizontal flight path it has left the atmosphere and does not waste fuel combating atmospheric drag.

Table 3 Initial and target parameters—launch to LEO

Specific impulse, s	450
Initial mass, kg	100,000
Thrust, N	1.6
Initial altitude, km	0
Initial velocity, m/s	1
Target altitude (LEO), km	372
Target velocity, m/s	7000
Final velocity (LEO), m/s	7684.5

The ΔV required for launch and LEO insertion for this run is 9530 m/s. If the Earth's rotation is included, a bonus of 408 m/s from a 28.5 deg-inclination launch site brings the actual ΔV down to 9122 m/s. This compares very well with commonly accepted launch ΔV requirements of 30,000 ft/s or 9144 m/s, and shows that the technique performs acceptably well for the difficult problem of generating a feasible launch trajectory.

Conclusions

The cases studied in the simulations demonstrate both the excellent targeting accuracy and good performance achievable with the orbit-fitting targeting technique. While, in some cases, an understanding of the technique is required and appropriate constraitns must be applied, the technique is basically easy to implement and use. It performs well for vehicles with very low thrust levels, and can be used to estimate fuel requirements when impulsive maneuvers are not permitted. The technique is ideally suited for use in applications such as design studies and mission feasibility studies where reasonable trajectories must be generated for many different choices of design parameters, but where the exact optimum trajectory is not needed. It also provides a good initial feasible trajectory for use with gradient-search-type optimization techniques that require such a trajectory as a starting point.

In addition, the orbit-fitting technique can be used to satisfy multiple choices of target parameter combinations, not just one designated set, thus making it more flexible than most current targeting methods. Because of its general nature, it could be applied to almost any two-dimensional, two-body, spaceflight targeting problem involving target parameters among those discussed.

Acknowledgments

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